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BPS orientifold planes from crosscap states in Calabi-Yau compactifications

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Abstract: We use the results of [1] on the simple current classification of open unoriented CFTs to construct half supersymmetry preserving crosscap states for rational Calabi-Yau compactifications. We show that the corresponding orientifold fixed planes obey the BPS-like relation $M = e^{\mathrm{i}\phi}Q$. To prove this relation, it is essential that the worldsheet CFT properly includes the degrees of freedom from the uncompactified space-time component. The BPS-phase ϕ can be identified with the automorphism type of the crosscap states. To illustrate the method we compute crosscap states in Gepner models with each k_i odd.

Keywords: D-branes, Superstring Vacua, Conformal Field Models in String Theory.

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Contents

1. Introduction

In this letter we present a universal class of crosscap states for $\mathcal{N} = 2$ rational CFTs corresponding to strings on Calabi-Yau manifolds. A generic property of a 'rational' Calabi-Yau compactification is that the worldsheet theory needs to be GSO projected to ensure space-time supersymmetry. Such a GSO projection is equivalent to a so called simple current extension by the spectral flow current S of the underlying chiral symmetry algebra of the worldsheet CFT. In [1] consistent boundary and crosscap states for arbitrary simple current modular invariants, of which simple current extensions are a subset, have been constructed. We can therefore apply the results of [1] to the specific case of a CY compactification. The connection between the simple current extension with S and space-time supersymmetry is reflected in that these simple current boundary and crosscap states preserve half the space-time supersymmetry. Half supersymmetric boundary states in CY compactifications, i.e. BPS D-branes, have been extensively studied in the past few years, both from the orbifold (see e.g. [2]–[4] and citations thereof) and simple current point of view (see e.g. [5]–[7]). Consistent type I CY compactifications with D-branes, however, need half-supersymmetry preserving orientifold planes. It is the corresponding crosscap states we discuss here.

In section 2 we will give an intuitive explanation of the results of [1] for $\mathbb{Z}_N$ simple current invariants. We will for simplicity assume that the currents do not have fixed points. In the presence of fixed points, the formulas for the boundary states change qualitatively [5, 6], whereas those for the crosscap states do not. Our focus is here on the latter and we believe that the inclusion of fixed points will not change our main results. In section 3 we then apply these formulas to 'rational' CY compactifications. We show that, when the uncompactified part of space-time is properly taken into account, the O-planes are BPS-like, i.e. they obey a mass-charge relation $M = e^{\mathrm{i}\phi}Q$, and we determine the phase ϕ in terms of CFT quantities. The BPS-like relation is a consequence of preserving half of the space-time supersymmetries. Interestingly, for D-branes the BPS property can be derived from the boundary state without reference to the uncompactified part of the theory [7]. We explain at the mathematical level why this is so. Finally we briefly illustrate these methods with the computation of the mass and charges of crosscap states in Gepner models with each k_i odd.

Despite the required presence of crosscap states in consistent type I CY compactifications, the interest in unoriented $\mathcal{N} = 2$ CFTs has begun rather recently [8]–[12]. Using a more geometrical approach based on linear sigma models, [10] found the locations of CY-orientifold planes as fixed points of anti-holomorphic [9, 11] or holomorphic isometries. To understand type I CY compactifications, we wish to know their physical characteristics, such as charge and tension, as well. At rational points in the CY moduli space, the method described here can be used to determine these.

In the midst of this project we noticed the posting of the conference proceedings [13] to the archive, in which BPS crosscap states for Gepner Models are constructed by exploiting the phase symmetries (see also the subsequent article [30]). Since phase symmetries are realized by simple currents, the results of [13] are in agreement with our results. The power of the construction proposed here is that it is applicable to any rational $N = 2$ SCFT that describes a CY compactification.

2. Boundary and crosscap states for simple current extensions

Consider a CFT with the same left- and right-chiral algebra $\mathcal{A}$ and a C-diagonal torus partition function (i.e. the modular invariant theory which pairs left-movers with right-moving Charge Conjugates). A complete set of boundary states that preserve a diagonal subalgebra of the left-right symmetry algebra $\mathcal{A} \times \bar{\mathcal{A}}$ is given by the Cardy states [14]

$$|B_a\rangle = \sum_i \frac{S_{ia}}{\sqrt{S_{i0}}} |i\rangle\rangle_1 \,, \tag{2.1}$$

where S_{ia} is the modular S-matrix of $\mathcal{A}$ and the sum is over all primaries. The Ishibashi state $|i\rangle\rangle_1$ is a coherent state of all C-diagonal closed string states in sector i. The boundary state preserves a diagonal subalgebra, which means that all states contribute with the same weight, namely 1, to $|i\rangle\rangle_1$. The boundary label a runs over all primaries of $\mathcal{A}$. Different labels can be thought of as labeling branes wrapping different cycles.

Fuchs and Schweigert extended this result of Cardy and constructed boundary states for theories whose modular invariant is a simple current extension [15]. Recall [16] that simple currents J are primary fields whose fusion with any other primary i field yields a *single* field $j = Ji$. Integer conformal weight simple currents can be used to extend the chiral algebra $\mathcal{A} \overset{J}{\subset} \mathcal{A}^{\rm ext}$. Under this extension primary fields arrange themselves into orbits $[i] = \{i, Ji, J^2 i, \ldots\}$. Orbits with integer monodromy charge $Q_J(i) \equiv h_i + h_J - h_{Ji} \bmod \mathbb{Z}$ under J, are the primaries of $\mathcal{A}^{\rm ext}$. Non-integer charged fields are projected out. Such an extension is often referred to as a 'simple current orbifold'. Although not quite correct from the worldsheet point of view,[1] this terminology sometimes makes sense from the point of view of the target space. In WZW models based on Lie group G for example, an extension by a simple current group $\mathbb{Z}_N$ amounts geometrically to strings moving on $G/\mathbb{Z}_N$.

With this geometric picture in mind, a natural guess for the boundary state of a $\mathbb{Z}_N$ extension is a sum over 'images',

$$|B_{[a]}\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |B_{J^n a}\rangle \,, \tag{2.2}$$

where J is a simple current that generates the $\mathbb{Z}_N$. (The normalization follows from CFT arguments, see eq. (2.8) below.) An important simple current identity for the modular S-matrix [16, 17],

$$S_{Ji,j} = e^{2\pi {\rm i} Q_J(j)} S_{i,j} \,, \tag{2.3}$$

allows the boundary state to be written as

$$|B_{[a]}\rangle = \sqrt{N} \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} \frac{S_{ia}}{\sqrt{S_{i0}}} \, |[i]\rangle\rangle_{Q_J(a)} \,. \tag{2.4}$$

Here the sum is over representatives of chargeless J-orbits and

$$|[i]\rangle\rangle_{Q_J(a)} = \sum_{n=0}^{N-1} e^{2\pi {\rm i} n Q_J(a)} |J^n i\rangle\rangle_1 \,. \tag{2.5}$$

This natural guess indeed corresponds to the boundary states constructed by Fuchs and Schweigert. Moreover, these boundary states are 'more general' than the Cardy state (2.1) in the following sense. Recall that the primaries $(i, Ji, J^2 i, ..., J^{N-1} i)$ with $Q_J(i) = 0$ group into one primary $[i]$ of the extended algebra $\mathcal{A}^{\rm ext}$. We see that on the boundary state $|B_{[a]}\rangle$ the closed strings in sector i and Ji are reflected with a relative phase $e^{2\pi {\rm i} Q_J(a)}$. From the point of view of the extended algebra $\mathcal{A}^{\rm ext} \times \bar{\mathcal{A}}^{\rm ext}$, the boundary state therefore does not respect a diagonal subalgebra, but a twisted one. The boundaries are said to obey the *twisted gluing condition*

$$[J_n - (-1)^{h_J} e^{-2\pi {\rm i} Q_J(a)} \bar{J}_{-n}] |B_{[a]}\rangle = 0 \,, \tag{2.6}$$

with respect to the simple current $J \in \mathcal{A}^{\rm ext}$. The phase $e^{-2\pi {\rm i} Q_J(a)}$ is the *automorphism type* of the boundary. Note that for a $\mathbb{Z}_N$ simple current J, the monodromy charge is a fraction of N:

$$Q_J(a) = \frac{n}{N} \,, \qquad n \in \mathbb{Z} \,. \tag{2.7}$$

[1] The difference emphasized in the introduction is that a worldsheet-orbifold makes a chiral algebra smaller, whereas an extension makes it larger. Orbifolds and simple current extensions are in fact each others inverse.

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Thus the automorphism type takes values in $\mathbb{Z}_N$. In particular for $Q_J(a) = 0 \bmod 1$ the boundary state (2.4) preserves the diagonal subalgebra. By construction this is the usual Cardy state for $\mathcal{A}^{\rm ext}$. We infer therefore — correctly — that the modular S-matrix of the extended theory is expressible in terms of the modular S-matrix of the original theory

$$S_{[a][b]} = N S_{ab}\,. \tag{2.8}$$

The N-dependence in this relation explains the normalization choice in eq. (2.2).

Pradisi, Sagnotti and Stanev [18] (PSS) found the formula analogous to Cardy's for crosscap states: [2]

$$|\Gamma\rangle_0^{\sigma(0)} = \sigma(0) \sum_i \frac{P_{i0}}{\sqrt{S_{i0}}} |i\rangle\rangle_{1,C}\,. \tag{2.9}$$

Here P is the pseudo-modular matrix $P = \sqrt{T} S T^2 S \sqrt{T}$ built from the modular T- and S-matrices, the sum runs over all primaries and 0 denotes the vacuum representation. In addition $\sigma(0)$ is the undetermined sign in the Möbius strip, which is ultimately fixed by tadpole cancellation. The crosscap Ishibashi state is similar to boundary Ishibashi states, except that even/odd levels contribute with opposite signs. A modified crosscap exists for every simple current K of $\mathcal{A}$, given by [20][3]

$$|\Gamma\rangle_K^{\sigma(K)} = \sigma(K) \sum_i \frac{P_{iK}}{\sqrt{S_{i0}}} |i\rangle\rangle_{1,C}\,. \tag{2.10}$$

The primary K is called the Klein bottle current (KBC). In WZW models [22] it was shown that the label K plays a role that is similar to the boundary label, namely different labels K represent O-planes at different locations. A natural guess for the crosscap state in simple current extensions is therefore [1, 23, 24]

$$|\Gamma\rangle_{[K]}^{[\sigma]} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |\Gamma\rangle_{J^n K}^{\sigma(J^n K)}\,. \tag{2.11}$$

As we will see the signs $\sigma(J^n K)$ are not completely arbitrary.

In the case of boundary states, the important identity (2.3) allowed us to 'perform' the sum over images. For the P-matrix, however, the analogous simple current identity is at first sight very different [24]

$$P_{J^2 i,j} = \epsilon_{J^2}(i) e^{2\pi \mathrm{i}[Q_J(j) - Q_J(Ji)]} P_{i,j}\,, \tag{2.12}$$

where

$$\epsilon_J(i) := e^{\pi \mathrm{i}[h_i - h_{Ji}]}\,. \tag{2.13}$$

[2] See [19] for a review.

[3] Note that the crosscap coefficient in [20] is written slightly different, namely $\Gamma'_i = P_{iK}/\sqrt{S_{iK}}$. In [21] it is argued that $\Gamma_i = P_{iK}/\sqrt{S_{i0}}$ is the correct crosscap state and that the Klein bottle is calculated as $K_m = \sum_i \Gamma_i \Gamma_i S_{im} e^{2\pi \mathrm{i} Q_K(i)}$, where $e^{2\pi \mathrm{i} Q_K(i)}$ is due to the action of Ω on the Ishibashi states. Similar remarks apply to the boundary state.

Most importantly, for N even, this relation is truly *qualitatively* different than its analogue (2.3). In that case only P-matrix elements on the same J-orbit that differ by *two* steps are related. From now on we focus on the interesting case N even. (This is also the case relevant for CY compactifications.)

Suppose first that the signs $\sigma(J^n K)$ in (2.11) are such that[4]

$$\begin{aligned} \sigma(J^{2m}K)\epsilon_{J^{2m}}(K)e^{-2\pi \mathrm{i} Q_{J^m}(J^m K)} &= \sigma_0 \,, \\ \sigma(J^{2m+1}K)\epsilon_{J^{2m}}(JK)e^{-2\pi \mathrm{i} Q_{J^m}(J^m JK)} &= \sigma_1 \,, \end{aligned} \tag{2.14}$$

for any pair of signs σ_0, σ_1. As we show in appendix A, these choices ensure that only $Q_J(i) = 0$ primaries, i.e. fields of $\mathcal{A}^{\rm ext}$, couple to the crosscap. With this choice the crosscap state equals

$$|\Gamma\rangle^\sigma_{[K]} = \sqrt{N} \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} \left(\frac{\sigma_0 P_{iK} + \sigma_1 P_{i,KJ}}{2\sqrt{S_{i0}}} \right) |[i]\rangle\rangle_{\epsilon_J(K)\sigma} \,, \tag{2.15}$$

where $\sigma = \sigma_0/\sigma_1$ and

$$|[i]\rangle\rangle_{\epsilon_J(K)\sigma} = \sum_{n=0}^{N-1} [\epsilon_J(K)\sigma]^n \epsilon_{J^n}(i) |J^n i\rangle\rangle_{1,C} \,. \tag{2.16}$$

Therefore, the crosscap states obey a twisted gluing condition,

$$[J_n - (-1)^{n+h_J} \epsilon^*_J(K)\sigma \bar{J}_{-n}] |\Gamma\rangle^\sigma_{[K]} = 0 \,. \tag{2.17}$$

and $\epsilon^*_J(K)\sigma = (\epsilon_J(K)\sigma)^{-1}$ is the automorphism type of the crosscap state. Note that the sign $\epsilon_{J^n}(i) = e^{\pi \mathrm{i}[h_i - h_{J^n i}]}$ in (2.16) is the parity of the level of the descendant $J^n i$ in the module $[i]$. To make sure that the overall sign of the crosscap Ishibashi state $|[i]\rangle\rangle_{\epsilon_J(K)\sigma}$ is correct, we must insist that the representative i has lowest conformal weight (mod 2) in the orbit $(i, Ji, \ldots, J^{N-1}i)$. From now on we assume that this is the case.

Not any Klein bottle current is allowed. The current K must have monodromy charge

$$Q_J(K) = \frac{2p}{N} \,, \qquad p \in \mathbb{Z} \,, \tag{2.18}$$

otherwise (2.14) cannot be solved over signs [24]. As a consequence, $[\epsilon_J(K)\sigma]^N = 1$ and the automorphism type of the planes takes values in $\mathbb{Z}_N$. The automorphism types of simple current D-branes and orientifold planes are therefore the *same*, as we indeed geometrically expect, even though the responsible CFT mechanisms are different.

Suppose we would make sign choices in (2.11) other than (2.14). In that case only charged fields would couple to the crosscap. Charged fields are not representations of the simple current extended algebra, and there is no linear combination of J and $\bar{J}$ that is preserved by the crosscap. Such crosscap states are not obviously inconsistent, but beyond the scope of this letter and we will proceed with (2.14) and the result (2.15).

[4]Note that for an integer spin current $\epsilon^2_{J^{2m}}(K)e^{-4\pi \mathrm{i} Q_{J^m}(J^m K)} = e^{2\pi \mathrm{i}[Q_{J^{2m}}(K) - 2Q_{J^m}(J^m K)]} = 1$, therefore $\sigma_0,\ \sigma_1$ are indeed signs.

Again, when the automorphism type is trivial, i.e. $\epsilon_J(K)\sigma = 1$, we expect (2.15) to be a PSS+KBC crosscap state for the extended algebra $\mathcal{A}^{\rm ext}$ by construction. We infer that the P-matrix of a simple current extension is related to the original P-matrix as [23, 24]

$$P_{[i][j]} = \sum_{n=0}^{N-1} \epsilon_{J^n}(i) P_{J^n i,j} = \frac{N}{2}[P_{ij} + \epsilon_J(i) P_{Ji,j}]\,, \tag{2.19}$$

where we used eq. (2.12) in the last step. Indeed, this expression is correct when the representatives i and j are chosen to have the lowest conformal weight (mod 2) in their orbits (and the currents have no fixed points.) For later purposes we also need the P-matrix for non-cyclic simple current groups $\mathcal{G}$. The generalization is straightforward:

$$P_{[i][j]} = \sum_{J\in\mathcal{G}} \epsilon_J(i) P_{Ji,j}\,. \tag{2.20}$$

To conclude: we have shown that boundary and crosscap states of $\mathbb{Z}_N$ simple current invariants of $\mathcal{A}$ posses a $\mathbb{Z}_N$-automorphism type with respect to the current that extends $\mathcal{A}$ to $\mathcal{A}^{\rm ext}$. When N is even the boundary and crosscap states of the extension have a qualitatively different feature: the boundary coefficient of boundary $[a]$ is the boundary coefficient of a representative a. In contrast to this, the crosscap coefficient of $[K]$ is the sum of the crosscap coefficient with Klein bottle currents K and JK.

3. Crosscap states for Calabi-Yau compactifications

In the previous section we have reviewed the theory of boundary and crosscap states in simple current extension invariants. In string theory, simple currents play a role in at least three steps in the construction of realistic models: (i) simple currents implement field identification and selection rules in coset constructions, (ii) simple currents realize alignment of spin structures in tensor product CFTs and (iii) they implement GSO projections. We will apply the general theory outlined above to the last step in the construction, the GSO projection.[5]

3.1 The bulk theory

We start by reviewing the closed string sector of Calabi-Yau compactifications in the language of CFT. We should stress that this construction is only appropriate at rational points of the CY moduli space. These are 'very symmetric' points in the space of CY deformations where the $2d$ symmetry algebra is enlarged to a theory with only a finite number of primary fields (with respect to the Virasoro plus extended algebra). Examples are the Fermat polynomial representations of Calabi-Yau manifolds as vanishing loci in weighted projected space.

[5]The reason we only apply the theory to the last step is that $\mathcal{N} = 2$ worldsheet susy does not allow boundary or crosscap states with non-trivial automorphism type in alignment or identification currents. Hence the Cardy-PSS solution is sufficient for the first two extensions [5].

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The chiral algebra at a rational point of a type II compactification on a CY n-fold can be constructed as follows. The starting point in the construction of the worldsheet theory is a tensor product

$$D_{8-n,1} \otimes \mathcal{A}_{3n} \,, \tag{3.1}$$

where $D_{r,1}$ is the affine algebra based on SO$(2r)$ at level one and $\mathcal{A}_{3n}$ any rational chiral bosonic subalgebra of a $\mathcal{N} = 2$ superconformal algebra. The latter contains a Virasoro algebra with conformal anomaly $3n$ and a U$(1)_R$ algebra and has two distinguished simple currents. The *supercurrent* v has order 2 and spin $h_v = 3/2$. States with integer (half-integer) monodromy charge with respect to the supercurrent are in the NS (R) sector. The second simple current is called *spectral flow* s. It has spin $h_s = c/24 = n/8$ and model dependent order N_s. The monodromy charge $Q_s(\lambda) = q_\lambda/2 \bmod 1$ equals half the U$(1)_R$ charge q_λ of λ.

The $D_{8-n,1}$ factor in (3.1) describes the uncompactified part of the theory. The use of the unitary group SO$(16-2n)$ rather the non-unitary group SO$(10-2n, 2)$, corresponding to the Lorentz group of the space-time fermions plus bosonized superghosts, is called *the bosonic string map* [25] (for a review see [26]). The $D_{8-n,1}$ theory has four primaries, $X_{8-n} \in (O_{8-n}, S_{8-n}, V_{8-n}, C_{8-n})$ with conformal weights $(0, (8-n)/8, 1/2, (8-n)/8)$ that realize a simple current group $\mathbb{Z}_2 \times \mathbb{Z}_2$ when n is even and $\mathbb{Z}_4$ when n is odd. The vector V_{8-n} plays the role of the supercurrent. The singlet and vector are in the NS sector and the spinor and conjugate spinor are in the R sector. In order to read off the string spectrum from the partition functions, one has to perform the (inverse) bosonic string map [25]:

$$\{O_{8-n}, S_{8-n}, V_{8-n}, C_{8-n}\} \to \{V_{4-n}, -S_{4-n}, O_{4-n}, -C_{4-n}\} \,. \tag{3.2}$$

Note that the current (S_{8-n}, s) has conformal weight 1 and can therefore be used to extend the chiral algebra. This is one of the reasons for using SO$(16-2n)$ instead of the expected little group SO$(8-2n) \subset$ SO$(10-2n, 2)$.

The algebra (3.1) cannot describe a $\mathcal{N} = 2$ superconformal theory since the spin structures (R or NS) of the space-time and internal part are not aligned. A superconformal theory is obtained when we extend (3.1) by $W \equiv (V_{8-n}, v)$. Let us denote this extended algebra by $\mathcal{A}^{ws}$. The primaries of $\mathcal{A}^{ws}$ are $[X_{8-n}, \lambda]$ and are subject to the following identification and selection rules:

$$[X_{8-n}, \lambda] \sim [V_{8-n} X_{8-n}, v\lambda] \,, \qquad Q_v(\lambda) = Q_{V_{8-n}}(X_{8-n}) \bmod 1 \,. \tag{3.3}$$

By the identification, the order N_S of the spectral flow current $S \equiv [S_{8-n}, s]$ is either N_s or $N_s/2$, depending on the model under consideration. The supercurrent of $\mathcal{A}^{ws}$ is $V \equiv [O_{8-n}, v]$ and the vacuum is $O \equiv [O_{8-n}, 0]$. Note however that $N_S \in 2\mathbb{Z}$ because $S_{8-n}^2 = O_{8-n}$ for n even and $S_{8-n}^4 = O_{8-n}$ for n odd.

We can use the spectral flow current to build a simple current modular invariant $\mathcal{Z}(\mathcal{A}^{ws}, S)$. This procedure is analogous to the GSO projection and the resulting CFT describes closed oriented strings on $\mathbb{R}^{1,9-n} \otimes CY_n$. This CFT has fields $S(z)$ and $\bar{S}(\bar{z})$ whose zero modes S_0 and $\bar{S}_0$ generate space-time supersymmetry transformations. Because S has integer spin, the invariant $\mathcal{Z}(\mathcal{A}^{ws}, S)$ can be thought of as a C-diagonal invariant of a larger algebra $\mathcal{A}^{\rm ext}$ obtained from $\mathcal{A}^{ws}$ by an extension by S. This section can therefore be summarized by the following sequence of embeddings

$$D_{8-n,1}\otimes\mathcal{A}_{3n}\overset{(V_{8-n},v)}{\subset}\mathcal{A}^{ws}\overset{S}{\subset}\mathcal{A}^{\rm ext}\,. \tag{3.4}$$

where we have indicated which simple currents are used as extensions. Note that this split-up is different from the one used in [5]. The guiding philosophy there was that the properties of branes on the CY manifold, i.e. boundary states in the internal CFT should be quite independent of the external space-time. This led the authors to first extend $\mathcal{A}_{3n}$ ($\mathcal{A}^{wsusy}$ in their notation) by the current $u\equiv v^n s^2$ to $\mathcal{A}^{cy}$; a half-way GSO-projection onto integer $\mathrm{U}(1)_R$ charges (integer worldsheet fermion number). The latter theory $\mathcal{A}^{cy}$ contains the fields o_{cy},s_{cy}, v_{cy} and c_{cy} — all simple currents — with the same fusion rules as the primary fields of $D_{8-n,1}$. A second extension by these currents yields $\mathcal{A}^{\rm ext}$. Because the latter extension acts without fixed points, the boundary states of $\mathcal{A}^{cy}$ and $\mathcal{A}^{\rm ext}$ indeed have the same qualitative features. However this latter extension is a $\mathbb{Z}_{\rm even}$ extension and therefore the crosscap states of $\mathcal{A}^{cy}$ and $\mathcal{A}^{\rm ext}$ are qualitatively different, as explained in the previous section. In particular, the crosscap state of $\mathcal{A}^{\rm ext}$ is the sum of the PSS crosscap state and the PSS+KBC crosscap state of $\mathcal{A}^{cy}$, where the Klein bottle current is S. We will see this in more detail in the next subsection.

3.2 A BPS condition for O-planes

Using the general theory reviewed in section 2, it is now straightforward to write down a consistent set of crosscap states for rational CY compactifications. The theory $\mathcal{A}^{\rm ext}$, describing strings on CY and obtained from $\mathcal{A}^{ws}$ by extension with S, has crosscap states

$$|\Gamma\rangle^{\sigma}_{[K]}=\sqrt{N_S}\sum_{\{i|REP_{[i]},\ Q_S(i)=0\}}\left(\frac{\sigma_0 P_{iK}+\sigma_1 P_{i,KS}}{2\sqrt{S_{0i}}}\right)|[i]\rangle\rangle_{\epsilon_S(K)\sigma}\,. \tag{3.5}$$

where K a simple current of $\mathcal{A}^{ws}$ and $\sigma=\sigma_0/\sigma_1$. For $K=O$ we see indeed that this crosscap state is the sum of a PSS crosscap state and the PSS+KBC crosscap state, where the Klein bottle current is S. This crosscap state can therefore have non-trivial automorphism type with respect to the spectral flow current S. With respect to all other currents of $\mathcal{A}^{\rm ext}\times\bar{\mathcal{A}}^{\rm ext}$ it preserves the *diagonal* subgroup. One of these currents is the $\mathrm{U}(1)_R$ current. The corresponding orientifold-planes are therefore all of A-type [27, 10].

From this formula we can derive some basic properties of the corresponding orientifold fixed planes. One should keep in mind that a single crosscap state, specified by the triple (K,σ_0,σ_1) represents in general a configuration of O-planes (WZW models are an example [22]).

- The gluing condition (2.17) implies that these planes preserve one half of space-time supersymmetry generated by the linear combination $S_0+\epsilon_S^*(K)\sigma\bar{S}_0$.
- The O-planes are BPS-like. From the expansion (3.5) one can read off the charges of the O-planes with respect to closed strings $|j;\bar{j}\rangle$ by calculating the overlap $\langle j;\bar{j}|\Gamma\rangle^{\sigma}_{[K]}$. The mass M and central charge Q of the planes are given by

$$M=\langle O;\bar{O}|\Gamma\rangle^{\sigma}_{[K]}\,,\qquad Q=\langle S;\bar{S}|\Gamma\rangle^{\sigma}_{[K]}\,, \tag{3.6}$$

where $\langle O;\bar{O}|$ denotes the graviton and $\langle S;\bar{S}|$ is the top RR form [7]. This is the chiral-chiral outstate $\langle S;\bar{S}|$ with both left- and right $\mathrm{U}(1)_R$ charge maximal, $q_{\mathrm{U}(1)_R} = c/6$, and is obtained by acting on the vacuum with precisely the simple currents S and $\bar{S}$. From the formula for the twisted Ishibashi states (2.16) we then immediately infer that the planes obey a BPS-like relation

$$M = \sigma\epsilon_S(K)(-1)^{h_S}Q\,, \qquad \epsilon_S(K)(-1)^{h_S} = e^{\pi \mathrm{i}(h_S+h_K-h_{SK})} = e^{\pi i Q_S(K)}\,, \tag{3.7}$$

where we used that h_S is integer and $\sigma := \sigma_0\sigma_1$. Thus, the phase of the central charge is thus given by minus the automorphism type of the crosscap state (since $h_S = 1$).

Let us discuss the BPS-relation in some more detail. Explicitly the mass and the charge equal, up to a common normalization

$$M = \sigma_0\left(P_{K,O} + \sigma P_{SK,O}\right), \qquad Q = \sigma_0\left(P_{K,S} + \sigma P_{SK,S}\right). \tag{3.8}$$

Next we note that the identity (2.12) implies that

$$P_{NS,R} = 0 \tag{3.9}$$

for any superconformal algebra $\mathcal{A}$, as can be seen by taking for J the supercurrent V. Because $S \in R$ we therefore have

$$M = \begin{cases} \sigma_0 P_{OO} & \text{for} \quad K \in NS \\ \sigma_0\sigma P_{O,KS} & \text{for} \quad K \in R \end{cases}, \qquad Q = \begin{cases} \sigma_0\sigma P_{SK,S} & \text{for} \quad K \in NS \\ \sigma_0 P_{K,S} & \text{for} \quad K \in R \end{cases}. \tag{3.10}$$

Note that the value of M depends on the spin structure of K. The signs σ_0, σ determine whether the O-planes are O^+ or O^- planes.[6] For instance, when $K \in NS$ and $P_{00} > 0$, $\sigma_0 = -1$ represents O^+-planes and $\sigma_0 = 1$ represents O^--planes. Also note that, for a fixed choice of σ_0 with $K \in NS$, a sign flip in σ changes O-planes in anti-O-planes (and vice versa for $K \in R$).

We wish to emphasize that it is the proper inclusion of the uncompactified space-time theory that guarantees the BPS property. In other words, a single crosscap state of the $D_{8-n,1} \times \mathcal{A}^{cy}$ theory of [5] defined in the end of section 3, where the GSO projection is only halfway implemented, is *not* BPS. Let $u \equiv v^n s^2$ again denote the simple current that extends $\mathcal{A}_{3n}$ to $\mathcal{A}^{cy}$. A crosscap state of this extension is

$$|\tilde{\Gamma}\rangle = \sqrt{N_u} \sum_{\{i|REP_{[i]},Q_u(i)=0\}} \left(\frac{P_{i0} + P_{i,u}}{2\sqrt{S_{0i}}}\right) |[i]\rangle\rangle_{\epsilon_u(0)\sigma}\,, \tag{3.11}$$

where i are primaries of $D_{8-n,1} \times \mathcal{A}^{CY}$. (There are more crosscap state due to sign choices and KBC's. The point we want to make applies to all of them.) The mass equals $M = \langle 0;\bar{0}|\tilde{\Gamma}\rangle$ and the charge equals $Q = \langle S;\bar{S}|\tilde{\Gamma}\rangle$. Because $u \in NS$ we have $Q = 0$ due to (3.9).

[6]We adopt the odd convention that an O^+ plane has negative tension and a O^- plane has positive tension.

Since M generically nonzero, the BPS condition is violated. Another crosscap state is

$$|\tilde{\Gamma}\rangle_S = \sqrt{N_u} \sum_{\{i|REP_{[i]},Q_u(i)=0\}} \left(\frac{P_{iS} + P_{i,Su}}{2\sqrt{S_{0i}}} \right) |[i]\rangle\rangle_{\epsilon_u(S)\sigma} . \tag{3.12}$$

where we used S as a KBC. Now the mass vanishes and the charge does not. In a sense, the sum of (3.11) and (3.12) obey a BPS condition. (this is the proposal made in [13]. A similar proposal for minimal models was made in [10]) This sum is precisely what we get in (3.5), when we extend the theory $\mathcal{A}^{ws}$ by S, *including* its action on the uncompactified space-time part.

With our knowledge of simple current extensions, it is also straightforward to see why the theory $D_{8-n,1} \times \mathcal{A}^{cy}$ does yield BPS D-branes. The reason is that the boundary labels, i.e. the modular S-matrix, of the extended theory $\mathcal{A}^{\text{ext}} \supset D_{8-n,1} \times \mathcal{A}^{cy}$ is given by a representative of the S-matrix of $D_{8-n,1} \times \mathcal{A}^{cy}$. This is the content of eq. (2.4). Therefore all properties of the boundary states are indeed encoded in $\mathcal{A}^{cy}$. However as the extension is by an even current, the P-matrix is *not* given by a representative, the consequences of which we have explicitly shown and discussed above.

4. Example: A-type orientifolds of Gepner models

When $\mathcal{A}_{3n}$ is based on $\mathcal{N} = 2$ minimal models the invariant $\mathcal{Z}(\mathcal{A}^{ws}, S)$ is called a Gepner model. To illustrate the details of the computation of BPS CY-crosscaps and how the BPS relation arises, we compute the explicit mass and charge properties of a crosscap state in a Gepner model example. Recall that an $\mathcal{N} = 2$ minimal model $\mathcal{A}_k$ at level k has a representation as a $\mathrm{SU}(2)_k \times \mathrm{U}(1)_4/\mathrm{U}(1)_{2h}$ WZW coset; $h \equiv k + 2$. It has primaries (l, m, s) with $l = 0, \ldots, k$, $m = -h + 1, \ldots, h \bmod 2h$, $s = -1, \ldots, 2 \bmod 4$, subject to field identification $(l, m, s) = (k - l, m + h, s + 2)$ and selection rule $l + m + s \in 2\mathbb{Z}$. Field identification in fact corresponds to extension by the identification current $I = (k, h, 2)$ on the unconstrained primaries (l, m, s) [28]. The supercurrent is $v = (0, 0, 2)$ and spectral flow is realized by $s = (0, 1, 1)$. Two other noteworthy simple currents are the *phase symmetries* generated by $(0, 2, 0)$ of order h, and the current $(0, k + 2, 2)$; the latter is the only current that has fixed points and then only when k is even [6]. We proceed stepwise. First we construct the P-matrix of an $\mathcal{N} = 2$ minimal model. We extend the tensor products of minimal models to a $\mathcal{N} = 2$ superconformal theory $\mathcal{A}_{3n}$. Finally we apply the lessons from the previous section to compute the PSS O-plane mass and charge for odd k_i Gepner models.

4.1 The P-matrix in minimal models

Using that field identification can be viewed as a simple current extension, we can express the modular P matrix of a coset CFT G/H in terms of those of the G and H theories with the use of equation (2.19). There is one subtlety here. In general we only know the conformal weights in the coset modulo integers. We can compensate for our ignorance by the introduction of signs [29]

$$a_{(l,m,s)} := e^{\pi \mathrm{i}[h^{\text{true}}_{(l,m,s)} - (h_l - h_m + h_s)]} , \tag{4.1}$$

where

$$h_l = \frac{l(l+2)}{4h} \qquad h_m = \frac{m^2}{4h}\,, \qquad h_s = \frac{s^2}{8}\,. \tag{4.2}$$

For most explicit models, the signs a are known. For minimal model cosets they can be found in for instance [10].

With these compensatory signs, the P matrix of a minimal model at level k is

$$P^{\min,k}_{(l,m,s)(l',m',s')} = a_{(l,m,s)}a_{(l',m',s')}\sum_{n=0,1}\epsilon_{k^n}(l)\epsilon^*_{h^n}(m)\epsilon_{2^n}(s)P^{\mathrm{SU}(2)_k}_{l+n(k-2l),l'}\left[P^{\mathrm{U}(1)_{2h}}_{m+nh,m'}\right]^* P^{\mathrm{U}(1)_4}_{s+n2,s'}\ ; \tag{4.3}$$

the basic WZW P-matrices are

$$\begin{aligned} P^{\mathrm{SU}(2)_k}_{l,l'} &= \frac{1}{\sqrt{h}}\sin\left(\frac{\pi(l+1)(l'+1)}{2h}\right)\sum_{u=0}^{1}(-1)^{u(k+l+l')}\,, \qquad h=k+2 \\ P^{\mathrm{U}(1)_{2p}}_{m,m'} &= \frac{e^{-\frac{\pi i m m'}{2p}}}{2\sqrt{p}}\sum_{u=0}^{1}(-1)^{u(p+m+m')}\,. \end{aligned} \tag{4.4}$$

4.2 The P-matrix of $\mathcal{A}_{3n}$

Next we construct the tensor product

$$\mathcal{A}^{\mathrm{ten}}_{3n} := \mathcal{A}_{k_1}\otimes\cdots\mathcal{A}_{k_r} \tag{4.5}$$

with central charge $c = \sum_i 3k_i/(k_i+2) = 3n$. We will take all k_i odd to avoid fixed point ambiguities (though fixed points should have no qualitative effect on the crosscap states) and label the primaries by $(\vec{l};\vec{m};\vec{s}) := ((l_1,m_1,s_1),(l_2,m_2,s_2),\ldots,(l_r,m_r,s_r))$. The P-matrix of this tensor product is simply the product of the minimal model P-matrices

$$P^{\mathrm{ten}}_{(\vec{l};\vec{m};\vec{s}),(\vec{l}';\vec{m}';\vec{s}')} = \prod_{i=1}^{r}\left[P^{\min,k_i}_{(l_i,m_i,s_i),(l'_i,m'_i,s'_i)}\right]\,. \tag{4.6}$$

The currents $w_i = (v_1,0,\ldots,0,v_i,0,..0)$, $i=2,\ldots,r$ have integer spin and can be used to extend $\mathcal{A}^{\mathrm{ten}}_{3n}$ to a $\mathcal{N}=2$ superconformal algebra $\mathcal{A}_{3n}$. We denote its primaries by $[\vec{l};\vec{m};\vec{s}]$. The modular matrices of $\mathcal{A}_{3n}$ can be expressed in terms of those of the minimal models using the rules (2.8) and (2.20) above. In particular, the P-matrix of $\mathcal{A}_{3n}$ is [7]

$$P^{3n}_{[\vec{l};\vec{m};\vec{s}],[\vec{l}';\vec{m}';\vec{s}']} = \sum_{\substack{\vec{p} \\ p_j=0,1;\ j=2,\ldots,r}}\epsilon_{w_2^{p_2}\ldots w_r^{p_r}}((\vec{l};\vec{m};\vec{s}))P^{\mathrm{ten}}_{(w_2^{p_2}\cdots w_r^{p_r}(\vec{l};\vec{m};\vec{s})),(\vec{l}';\vec{m}';\vec{s}')}\,. \tag{4.7}$$

For notational purposes it is convenient to define

$$p_1 := \begin{cases} 0 \text{ for } \sum_{i=2}^r p_i & \text{even} \\ 1 \text{ for } \sum_{i=2}^r p_i & \text{odd}\,. \end{cases} \tag{4.8}$$

[7] This expression is correct when the arguments have the lowest conformal weight modulo 2 in the orbit. See the remarks around equation (2.19).

Writing out all the phases $a_{(l,m,s)}$ and $\epsilon_{\vec{\omega}}(\vec{l},\vec{m},\vec{s})$, the total phase simplifies and one finds for the P-matrix of the $\mathcal{A}_{3n}$ theory

$$
\begin{aligned}
P^{3n}_{[\vec{l};\vec{m};\vec{s}],[\vec{l}';\vec{m}';\vec{s}']} = \sum_{\substack{\vec{n},\vec{p} \\ n_i=0,1;\; i=1,\dots,r; \\ p_j=0,1;\; j=2,\dots,r.}} & \left[\prod_{i=1}^{r}\left(a_{(l_i',m_i',s_i')}\right)\phi_{n_i,p_i}((l_i,m_i,s_i))\times\right. \\
& \left.\times P^{\mathrm{SU}(2)_{k_i}}_{l_i+n_i(k_i-2l_i),l_i'}\left[P^{\mathrm{U}(1)_{2h_i}}_{m_i+n_ih_i,m_i'}\right]^* P^{\mathrm{U}(1)_4}_{s_i+2n_i+2p_i,s_i'}\right],
\end{aligned}
$$

with the phase

$$
\begin{aligned}
\phi_{n_i,p_i}((l_i,m_i,s_i)) &\equiv \epsilon_{\omega_i^{p_i}}((l_i,m_i,s_i))a_{(l_i,m_i,s+2p_i)}\epsilon_{k_i^{n_i}}(l_i)\epsilon^*_{h_i^{n_i}}(m_i)\epsilon_{2_i^{n_i}}(s_i+2p_i) \\
&= e^{\pi i\left[h^{\mathrm{true}}_{(l_i,m_i,s_i)}-h_{l_i+n_i(k-2l_i)}+h_{m_i+n_ih_i}-h_{s_i+2n_i+2p_i}\right]}. \qquad (4.9)
\end{aligned}
$$

4.3 Masses and charges of O-planes in $\mathcal{A}^{ws}_{\text{Gepner}}$

Using this explicit expression for the P-matrix of the $\mathcal{A}_{3n}$ built from minimal models, we can calculate the mass M and charge Q of the O-planes for odd level Gepner models. We have explained in detail, that a BPS relation is only obtained after (i) we tensor the $\mathcal{A}^{3n}$ theory with the space-time part described by the $D_{8-n,1}$ model, and (ii) this tensor product is extended by the vector current $W\equiv(V_{8-n},v_{3n})$ to $\mathcal{A}^{ws}$. For concreteness, we consider the PSS-crosscap state, i.e. we choose a trivial Klein bottle current. To find the charges and tension of the PSS-orientifold plane we only need to know the entries P^{ws}_{OO} and P^{ws}_{SS}. For a Gepner model built on r minimal models with each k_i odd, i.e. models without fixed points, the computation is straightforward and given in appendix B. One finds that these entries of the P-matrix are given by

$$
\begin{aligned}
P^{ws}_{OO} &= P^D_{00}P^{3n}_{00}+\epsilon_V(0)P^D_{0v}P^{3n}_{0v} \\
&= 2^{r/2}\cos\left(\frac{r\pi}{4}\right)\left(P^D_{00}+\tan\left(\frac{r\pi}{4}\right)P^D_{0v}\right)\prod_{i=1}^{r}\left(P^{\min,k_i}_{00}\right), \\
P^{ws}_{S,S} &= P^D_{s,s}P^{3n}_{s,s}+\epsilon_V(s)P^D_{s,vs}P^{3n}_{s,v_1s} \\
&= 2^{r/2}\cos\left(\frac{r\pi}{4}\right)\left(P^D_{s,s}-i\tan\left(\frac{r\pi}{4}\right)P^D_{s,vs}\right)\prod_{i}\left(P^{\min,k_i}_{ss}\right). \qquad (4.10)
\end{aligned}
$$

By (3.9), the mixed NS, R entry $P^{ws}_{O,S}$ of vanishes: $P^{ws}_{O,S}=0$. The non-zero entries of the P-matrix of $D_{8-n,1}$ are readily computed (see for instance [19])

$$
\begin{aligned}
P^D_{0,0} &= -P^D_{v,v}=\cos\left(\frac{n\pi}{4}\right), \\
P^D_{0,v} &= -\sin\left(\frac{n\pi}{4}\right), \\
P^D_{s,s} &= P^D_{c,c}=e^{\frac{in\pi}{4}}\cos\left(\frac{n\pi}{4}\right), \\
P^D_{s,c} &= -\mathrm{i}e^{\frac{in\pi}{4}}\sin\left(\frac{n\pi}{4}\right). \qquad (4.11)
\end{aligned}
$$

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Substituting these values, we get

$$P^{ws}_{O,O} = 2^{r/2} \cos\left(\frac{(n+r)\pi}{4}\right) \prod_{i=1}^{r} \left(P_{00}^{\min,k_i}\right), \tag{4.12}$$

$$P^{ws}_{S,S} = 2^{r/2} \cos\left(\frac{(n+r)\pi}{4}\right) e^{\frac{in\pi}{4}} \prod_{i=1}^{r} \left(P_{ss}^{\min,k_i}\right) = 2^{r/2} \cos\left(\frac{(n+r)\pi}{4}\right) \prod_{i=1}^{r} \left(P_{00}^{\min,k_i}\right),$$

upon using (B.5) and $n = \sum k_i/(k_i+2)$.

As $P^{ws}_{S,S}$ *equals* $P^{ws}_{O,O}$ we immediately see that the PSS-crosscap state has equal magnitude mass and central charge:

$$\begin{aligned} M &= \langle O|C\rangle^{+}_{O} = \sigma_0 P_{O,O}\,, & Q &= \langle S|C\rangle^{+}_{O} = \sigma_0 P_{S,S}\,, \\ M &= \langle O|C\rangle^{-}_{O} = \sigma_0 P_{O,O}\,, & Q &= \langle S|C\rangle^{-}_{O} = -\sigma_0 P_{S,S}\,. \end{aligned} \tag{4.13}$$

The BPS-like equality is verified, and the different choices for the automorphism sign σ are explicitly seen to correspond to planes vs. anti-planes. The result, of course, is an explicit manifestation of the identity (2.12) when we recall that $h_S = 1$.

5. Conclusions

The results of [1] have opened the way for a systematic study of a large class of open unoriented rational CFTs. Of special phenomenological interest are those unoriented RCFTs which correspond to type I string compactifications on 'rational' Calabi-Yau manifolds. These theories provide a new class of $\mathcal{N}=1$, $d=4$ string vacua, in addition to heterotic strings on Calabi-Yau 3-folds and M-theory on G_2 manifolds. In RCFT-based string theories, spacetime supersymmetry is implemented through a GSO projection. This projection can equivalently be viewed as a simple current extension by the spectral flow current S. Using the general theory of boundary and crosscap states for simple current extensions [1, 24], we have constructed here the crosscap states for such unoriented type I compactifications. These boundary and crosscap states are expressed in terms of the (pseudo-) modular matrices S and P, which are explicitly known for many CFTs, e.g. WZW models and cosets thereof.[8]

In particular, we have shown how the rational CY crosscap states (2.15) correspond to half-supersymmetry preserving A-type orientifold planes that are BPS-like. Their masses and central charges are equal up to a phase: $M = e^{i\phi}Q$, and this relation is a reflection of the simple current identity (2.12) obeyed by the modular P-matrix. The phase ϕ, moreover, is minus the automorphism-type of the crosscap state with respect to the spectral flow current. This BPS condition only holds when the uncompactified spacetime degrees of freedom are properly included in the GSO projection. This is in contrast with D-branes [7], where the BPS property follows from considering the internal sector independent of the space-time sector.

[8] Since the conformal weights h of coset theories are in general only known up to integers, the coset P-matrix is in general only known up to a sign.

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This study provides a step towards the classification of the orientifolds of a given Calabi-Yau manifold. RCFT methods are limited to CY manifolds at rational points in the moduli space, and ultimately one wishes for a geometric description where one can freely move away from 'rational' Calabi-Yaus. Progress towards a geometric formulation of orientifolds is in the early stages [8, 9, 11, 12]. A recent study of unoriented linear sigma models [10] showed that orientifold planes are located at fixed points of holomorphic or anti-holomorphic isometries. A next item is to determine their charges and tension. Matching with RCFT data, as obtained with the methods described here, can provide these.

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A. Crosscap states in simple current extensions

To show explicitly some of the properties of the crosscap state of a simple current extension, we perform here the steps discussed in section 2 in detail. Starting with the natural guess eq. (2.11) for a $\mathbb{Z}_{2m}$ simple current crosscap state (for ease of notation we have absorbed the normalization $1/\sqrt{2mS_{0i}}$ in the crosscap Ishibashi state)

$$|\Gamma\rangle^{[\sigma]}_{[K]} = \sum_{n=0}^{2m-1} \sum_i \sigma(J^n K) P_{i,J^n K} |i\rangle\rangle_{1,C} \,, \tag{A.1}$$

we use the simple current identity (2.12) for even and odd powers J^n respectively to obtain

$$\begin{aligned} |\Gamma\rangle^{[\sigma]}_{[K]} &= \sum_i \sum_{n=0}^{m-1} \left(\sigma(J^{2n}K) P_{i,J^{2n}K} + \sigma(J^{2n+1}K) P_{i,J^{2n+1}K}\right) |i\rangle\rangle_{1,C} \\ &= \sum_i \sum_{n=0}^{m-1} \Big(\sigma(J^{2n}K)\epsilon_{J^{2n}}(K) e^{2\pi \mathrm{i}[Q_{J^n}(i)-Q_{J^n}(J^nK)]} P_{i,K} + \\ &\qquad + \sigma(J^{2n+1}K)\epsilon_{J^{2n}}(JK) e^{2\pi \mathrm{i}[Q_{J^n}(i)-Q_{J^n}(J^{n+1}K)]} P_{i,JK}\Big) |i\rangle\rangle_{1,C} \,. \end{aligned} \tag{A.2}$$

Making the sign choices advocated in eq (2.14),

$$\begin{aligned} \sigma(J^{2m}K)\epsilon_{J^{2m}}(K) e^{-2\pi \mathrm{i} Q_{J^m}(J^m K)} &= \sigma_0 \,, \\ \sigma(J^{2m+1}K)\epsilon_{J^{2m}}(JK) e^{-2\pi \mathrm{i} Q_{J^m}(J^{m+1} K)} &= \sigma_1 \,, \end{aligned} \tag{A.3}$$

we get

$$|\Gamma\rangle^{[\sigma]}_{[K]} = \sum_i (\sigma_0 P_{i,K} + \sigma_1 P_{i,JK}) \sum_{n=0}^{m-1} e^{2\pi \mathrm{i} Q_{J^n}(i)} |i\rangle\rangle_{1,C} \,. \tag{A.4}$$

The final sum over phases $e^{2\pi \mathrm{i} Q_{J^n}(i)}$ immediately shows that fields with monodromy charge $Q_J(i) = 2k/2m,\ k \neq 0$ do not couple to the orientifold plane. Nor do odd charged fields, for in that case both P-matrix entries vanish. For a $\mathbb{Z}_{2m}$ current the P-matrix obeys

$$\begin{aligned} P_{a,i} &= P_{J^{2m}a,i} \\ &= \epsilon_{J^{2m}}(a) e^{2\pi \mathrm{i}[Q^m_J(i) - Q^m_J(J^m a)]} P_{a,i} \\ &= e^{2\pi \mathrm{i} m [Q_J(i) - Q_J(a)]} P_{a,i} \,. \end{aligned} \tag{A.5}$$

We used in the second line again the identity (2.12); in the third line the definitions for $\epsilon_J(K)$ and $Q_J(K)$, the fact that J has integer weight, and that $Q_{J^p}(J^q i) = p(Q_J(i) + qQ_J(J)) \bmod 1$. From this identity it follows that a P-matrix element with an even and and odd charged entry vanishes: for $Q_J(i) = 2k+1/2m$ and $Q_J(a) = 2p/2m$, $P_{a,i} = -P_{a,i} = 0$. As the current K must be even in order that eq. (A.3) can be solved (see eq. 2.18), hence for odd charged fields both P-matrix entries $P_{i,K} = 0$ and $P_{i,JK}$ vanish. This establishes the claim that the sign choice made, ensures that the O-plane only couples to fields in the extension, i.e. fields with $Q_J(i) = 0 \bmod 1$.

We may therefore limit the sum over primaries to the sum over chargeless primaries with no penalty. The final step is to rewrite the sum over $\mathcal{A}$ primaries, as a sum over representatives of chargeless orbits $[i]_{Q_J(i)=0}$, the primaries of the extended theory:

$$\begin{aligned} |\Gamma\rangle^{[\sigma]}_{[K]} &= \sum_{\{i|Q_J(i)=0\}} (\sigma_0 P_{i,K} + \sigma_1 P_{i,JK}) |i\rangle\rangle_{1,C} \\ &= \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} \sum_{n=0}^{2m-1} (\sigma_0 P_{J^n i,K} + \sigma_1 P_{J^n i,JK}) |J^n i\rangle\rangle_{1,C} \,. \end{aligned} \tag{A.6}$$

Splitting the sum into even and odd parts, and using the P-matrix simple current identity once more, we find

$$\begin{aligned} |\Gamma\rangle^{[\sigma]}_{[K]} = \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} \sum_{n=0}^{m-1} \Big[& \sigma_0\, \epsilon_{J^{2n}}(i)\ e^{2\pi \mathrm{i}[Q_{J^n}(K) - Q_{J^n}(J^n i)]}\ P_{i,K} |J^{2n} i\rangle\rangle_{1,C} + \\ & + \sigma_0\, \epsilon_{J^{2n}}(Ji) e^{2\pi \mathrm{i}[Q_{J^n}(K) - Q_{J^n}(J^{n+1} i)]} P_{Ji,K} |J^{2n+1} i\rangle\rangle_{1,C} + \\ & + \sigma_1\, \epsilon_{J^{2n}}(i)\ e^{2\pi \mathrm{i}[Q_{J^n}(JK) - Q_{J^n}(J^n i)]} P_{i,JK} |J^{2n} i\rangle\rangle_{1,C} + \\ & + \sigma_1\, \epsilon_{J^{2n}}(Ji) e^{2\pi \mathrm{i}[Q_{J^n}(JK) - Q_{J^n}(J^{n+1} i)]} P_{Ji,JK} |J^{2n+1} i\rangle\rangle_{1,C} \Big] . \end{aligned} \tag{A.7}$$

An alternative form of the P-matrix simple current identity (2.12) is [24]

$$P_{Jd,b} = \epsilon^*_J(d) \epsilon_J(b) P_{d,Jb} \,. \tag{A.8}$$

With this we see that

$$
\begin{aligned}
P_{Ji,K} &= \epsilon_J^*(i)\epsilon_J(K)P_{i,JK}\,,\\
P_{Ji,JK} &= \epsilon_J^*(i)\epsilon_J(JK)P_{i,J^2K} = \epsilon_J^*(i)\epsilon_J(JK)\epsilon_{J^2}(K)e^{2\pi\mathrm{i}[Q_J(i)-Q_J(JK)]}P_{i,K}\\
&= \epsilon_J^*(i)\epsilon_J(K)e^{2\pi\mathrm{i}[Q_J(i)-h_J]}P_{i,K}\,,
\end{aligned}
\tag{A.9}
$$

where we used (2.12) and the definitions of $\epsilon_J(K)$ and $Q_J(K)$ in the second and third step. Using that the field i is chargeless: $Q_J(i)$ is zero, and that J is a simple current used in an extension, i.e. it has integer conformal weight, one can show that $\epsilon_J^*(i) = \epsilon_J(i)$. Substituting these identities above we find

$$
\begin{aligned}
|\Gamma\rangle_{[K]}^{[\sigma]} = \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} \sum_{n=0}^{m-1} \Big[&(\sigma_0 P_{i,K} + \sigma_1 P_{i,JK})\ \epsilon_{J^{2n}}(i) e^{2\pi\mathrm{i}Q_{J^n}(K)}\ \ |J^{2n}i\rangle\rangle_{1,C} +\\
&+ (\sigma_0\, P_{i,JK} + \sigma_1 P_{i,K})\epsilon_{J^{2n}}(Ji)\epsilon_J(i)\epsilon_J(K) \times\\
&\times e^{2\pi\mathrm{i}Q_{J^p}(K)} P_{i,JK}|J^{2n+1}i\rangle\rangle_{1,C}\Big]\,,
\end{aligned}
\tag{A.10}
$$

where we have again used that $Q_{J^p}(J^q i) = p(Q_J(i) + qQ_J(J)) \bmod 1$, and that $Q_J(J) = 0 \bmod 1$. Finally realizing that for an integer spin current J, $e^{2\pi i Q_J(K)} = \epsilon_J^2(K)$, and recombining of the explicit expressions for the various $\epsilon_J(i)$ terms, we can simplify the expression for the crosscap state to

$$
\begin{aligned}
|\Gamma\rangle_{[K]}^{[\sigma]} &= \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} (\sigma_0 P_{i,K} + \sigma_1 P_{i,JK}) \sum_{n=0}^{m-1} [\epsilon_J^{2n}(K)\epsilon_{J^{2n}}(i)|J^{2n}i\rangle\rangle_{1,C} +\\
&\qquad\qquad + \sigma\epsilon_J^{2n+1}(K)\epsilon_{J^{2n+1}}(i)|J^{2n+1}i\rangle\rangle_{1,C}]\\
&= \sum_{\{i|REP_{[i]},\ Q_J(i)=0\}} (\sigma_0 P_{i,K} + \sigma_1 P_{i,JK}) \sum_{n=0}^{2m-1} [\sigma\epsilon_J(K)]^n\,\epsilon_{J^n}(i)|J^n i\rangle\rangle_{1,C}\,.
\end{aligned}
\tag{A.11}
$$

This is eq. (2.15).

B. P-matrix entries in all k_i odd Gepner models

- P_{OO}^{ws} Recall the forms of the SU(2) and U(1) WZW P-matrices, eq. (4.4). Due to the selection rule $k + l_1 + l_2 \in 2\mathbb{Z}$ for the SU(2) P-matrix, the P_{00} and P_{0v} entries of the P-matrix of the odd k minimal models are given by a single term (recall the conformal weights $h_{k,h,2} = 0$, $h_{0,0,2} = h_{k,k+2,0} = 3/2$, see e.g. [10])

$$
\begin{aligned}
P_{00}^{\min} &= \frac{\sqrt{2}}{(k+2)} \sin\left(\frac{\pi(k+1)}{2k+4}\right),\\
P_{0v}^{\min} &= P_{00}^{\min}\,.
\end{aligned}
\tag{B.1}
$$

Tensoring r odd k_i minimal models and extending by w_i we obtain for the P_{00} and P_{0v} elements of the $\mathcal{A}^{3n}$ theory (use that $\epsilon_v(0) = e^{-3\pi\mathrm{i}/2}$)[9]

$$\begin{aligned}
P_{00}^{3n} &= \prod_{i=1}^{r}\left(P_{00}^{\mathrm{min},k_i}\right) + \sum_{i<j}\epsilon_{v_i}(0)P_{0v_i}\epsilon_{v_j}(0)P_{0v_j}\prod_{k\neq 1,j}\left(P_{00}^{k_k}\right) + \\
&\quad + \sum_{i<j<k<l}\epsilon_{v_i}(0)P_{0v_i}\epsilon_{v_j}(0)P_{0v_j}\epsilon_{v_k}(0)P_{0v_k}\epsilon_{v_l}(0)P_{0v_l}\prod_{n\neq i,j,k,l}\left(P_{00}^{k_n}\right) + \cdots \\
&= \sum_{p=0}^{r/2}(-1)^p\binom{r}{2p}\prod_{i=1}^{r}\left(P_{00}^{\mathrm{min},k_i}\right) \\
&= 2^{r/2}\cos\left(\frac{r\pi}{4}\right)\prod_{i=1}^{r}\left(P_{00}^{\mathrm{min},k_i}\right), \\
P_{0v}^{3n} &= \sum_{p=0}^{r/2}(-1)^p\binom{r}{2p+1}\prod_{i}\left(P_{00}^{\mathrm{min},k_i}\right) \\
&= 2^{r/2}\sin\left(\frac{r\pi}{4}\right)\prod_{i}\left(P_{00}^{\mathrm{min},k_i}\right).
\end{aligned} \tag{B.2}$$

Finally extending with W we get (note that $h_W = 2$, hence $\epsilon_W(O) = 1$)

$$\begin{aligned}
P_{OO}^{ws} &= P_{00}^{D}P_{00}^{3n} + \epsilon_W(0)P_{0v}^{D}P_{0v}^{3n} \\
&= 2^{r/2}\cos\left(\frac{r\pi}{4}\right)\left(P_{00}^{D} + \tan\left(\frac{r\pi}{4}\right)P_{0v}^{D}\right)\prod_{i=1}^{r}\left(P_{00}^{\mathrm{min},k_i}\right).
\end{aligned} \tag{B.3}$$

- $P_{O,S}^{ws}$ From the explicit expression

$$\begin{aligned}
P_{(\lambda,\mu,\sigma),(l,m,s)}^{\mathrm{min}} &= P_{\lambda,l}^{\mathrm{SU}(2)}\left(P_{\mu,m}^{\mathrm{U}(1)}\right)^{*}P_{\sigma,s}^{\mathrm{U}(1)} + \\
&\quad + e^{\pi\mathrm{i}[h_{l,m,s}-h_{k-l,m+h,s+2}]}P_{\lambda,k-l}^{\mathrm{SU}(2)}\left(P_{\mu,m+h}^{\mathrm{U}(1)}\right)^{*}P_{\sigma,s+2}^{\mathrm{U}(1)},
\end{aligned} \tag{B.4}$$

it is easy to see that $P_{0,s}^{\mathrm{min}} = 0$ for k odd. The selection rule that $k+a+b \in 2\mathbb{Z}$ for $\mathrm{SU}(2)_k$ P-matrices implies that in that case only the second term contributes, but the same selection rule $k+a+b \in 2\mathbb{Z}$ for the $\mathrm{U}(1)_{2k+4}$ factor then shows that $P_{00}^{\mathrm{min}} = 0$. Since the currents v_i only act in the $\mathrm{U}(1)_4$ sector, this immediately shows $P_{O,S}^{ws} = 0$. Of course, this is simply an example of $P_{NS,R} = 0$

- $P_{S,S}^{ws}$ For the entry P_{ss}^{min} again only the second term in (B.4) contributes due to the SU(2) selection rule

$$P_{(0,1,1),(0,1,1)}^{\mathrm{min}} = e^{\pi\mathrm{i}[h_{(0,1,1)}-h_{(k,-k-1,-1)}]}P_{0,k}^{\mathrm{SU}(2)}\left(P_{1,-k-1}^{\mathrm{U}(1)}\right)^{*}P_{1,-1}^{\mathrm{U}(1)} = e^{-\frac{\pi\mathrm{i}k}{4k+8}}P_{00}^{\mathrm{min}}. \tag{B.5}$$

[9] Compared to the expression above eq. (4.9), which computes P_{00}^{3n} in one step, we have first performed the sum over n to obtain the minimal model P-matrix, and then extended by the currents w_i. In this second step, we used that

$$\epsilon_{w_2^{p_2}\ldots w_r^{p_r}}(\vec{l};\vec{m};\vec{s}) = \prod_{i=1}^{r}\epsilon_{w_i^{p_i}}((l_i,m_i,s_i)),$$

which is only true if one has chosen the correct representative of the orbit. In general one has to be careful with the additional phases $a_{(l_i,m_i,s_i)}$.

We also need

$$P_{s,vs}^{\min} = -e^{-\frac{\pi i(3k+4)}{4k+8}} P_{00}^{\min} = i P_{s,s}^{\min} \,. \tag{B.6}$$

Extending with the currents w_i we find ($\epsilon_v(s) = -1$)

$$\begin{aligned}
P_{ss}^{3n} &= \prod_{i=1}^{r} \left(P_{ss}^{\min,k_i}\right) + \sum_{i<j} \epsilon_{v_i}(s) P_{s,v_i s} \epsilon_{v_j}(s) P_{s,v_j s} \prod_{k\neq i,j} \left(P_{ss}^{k_k}\right) + \\
&\quad + \sum_{i<j<k<l} \epsilon_{v_i}(s) P_{s,v_i s} \epsilon_{v_j}(s) P_{s,v_j s} \epsilon_{v_k}(s) P_{s,v_k s} \epsilon_{v_l}(s) P_{s,v_l s} \prod_{n\neq i,j,k,l} \left(P_{ss}^{k_n}\right) + \cdots \\
&= \sum_{p=0}^{r/2} (-1)^p \binom{r}{2p} \prod_i \left(P_{ss}^{\min,k_i}\right) \\
&= 2^{r/2} \cos\left(\frac{r\pi}{4}\right) \prod_i \left(P_{ss}^{\min,k_i}\right) , \\
P_{s,v_1 s}^{3n} &= i 2^{r/2} \sin\left(\frac{r\pi}{4}\right) \prod_i \left(P_{ss}^{\min,k_i}\right) \,.
\end{aligned} \tag{B.7}$$

Extending finally with W, we get, using $\epsilon_W(S) = -1$,

$$\begin{aligned}
P_{S,S}^{ws} &= P_{s,s}^{D} P_{s,s}^{3n} + \epsilon_W(s) P_{s,vs}^{D} P_{s,v_1 s}^{3n} \\
&= 2^{r/2} \cos\left(\frac{r\pi}{4}\right) \left(P_{s,s}^{D} - i \tan\left(\frac{r\pi}{4}\right) P_{s,vs}^{D}\right) \prod_i \left(P_{ss}^{\min,k_i}\right) .
\end{aligned} \tag{B.8}$$

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